Finite Math - Fall 2018

Lecture Notes - 10/11/2018

Homework

- Section 4.1 - 75, 77, 80
- Section 4.2 - 9, 10, 15, 16, 19, 20, 23, 24, 26, 30, 31, 32, 34, 50, 52, 54, 55, 56

We now want to look at the case when the system does not have one unique solution, but is either inconsistent or is consistent but dependent.

Example 1. *Solve the system*

$$\begin{array}{rcrcr} 2x & + & 6y & = & -3 \\ x & + & 3y & = & 2 \end{array}$$

Solution. *We begin by subtracting twice the second equation from the first*

$$\begin{array}{rcrcr} & & 0 & = & -7 \\ x & + & 3y & = & 2 \end{array}$$

The first equation has become $0 = -7$ which is obviously untrue. This is an example of what happens when the system is inconsistent*.*

Example 2. *Solve the system*

$$\begin{array}{rcrcr} x & - & \frac{1}{2}y & = & 4 \\ -2x & + & y & = & -8 \end{array}$$

Solution. *First, let's multiply the first equation by 2 to get rid of the fraction*

$$\begin{array}{rcrcr} 2x & - & y & = & 8 \\ -2x & + & y & = & -8 \end{array}$$

Now, if we add the two equations together, we get

$$0 = 0$$

which is always true. This means that the two equations are the same equation, just one is (maybe) multiplied by a constant. This is a consistent but dependent *system of equations. If we let $x = k$, where k is any real number, then we get that $y = 2k - 8$. So, for any k, $(k, 2k - 8)$ is a solution. In this case, the variable k is called a* parameter*.*

Example 3. *Solve the systems*

(a)

$$\begin{array}{rcrcr} 5x & + & 4y & = & 4 \\ 10x & + & 8y & = & 4 \end{array}$$

(b)

$$\begin{array}{rcrcr} 6x & - & 5y & = & 10 \\ -12x & + & 10y & = & -20 \end{array}$$

Solution.

(a) *No solution.*

(b) *For any real number k, a solution is* $\left(k, \frac{6}{5}k - 2\right)$.

Applications. There are a variety of applications of systems of equations. For a simple example, consider the following

Example 4. *Dennis wants to use cottage cheese and yogurt to increase the amount of protein and calcium in his daily diet. An ounce of cottage cheese contains 3 grams of protein and 15 milligrams of calcium. An ounce of yogurt contains 1 gram of protein and 41 milligrams of calcium. How many ounces of cottage cheese and yogurt should Dennis eat each day to provide exactly 62 grams of protein and 760 milligrams of calcium?*

Solution. *We begin by setting up an equation for the amount of protein consumed and another equation for the amount of calcium consumed. Suppose Dennis eats x ounces of cottage cheese and y ounces of yogurt. Since 1 ounce of cottage cheese contains 3 grams of protein we know that x ounces of cottage cheese will contain $3x$ grams of protein; likewise y ounces of yogurt contains y grams of protein. Since Dennis wants to consume a total of 62 grams of protein, we get the equation*

$$3x + y = 62$$

We similarly set up an equation for milligrams of calcium consumed: x ounces of cottage cheese has $15x$ milligrams of calcium and y ounces of yogurt has 41 milligrams of calcium, and Dennis wants to eat exactly 760 milligrams of calcium, so

$$15x + 41y = 760$$

Thus we have a system of equations to solve which will tell us exactly how much cottage cheese and yogurt Dennis should eat

$$\begin{array}{rcrcr} 3x & + & y & = & 62 \\ 15x & + & 41y & = & 760 \end{array}$$

If we multiply the top equation by -5 then add the two equations together, we get

$$36y = 450$$

giving

$$y = 12.5$$

Plugging this into the first equation gives us

$$3x + 12.5 = 62 \iff 3x = 49.5 \iff x = 16.5.$$

So, Dennis should eat 16.5 *grams of cottage cheese and* 12.5 *grams of yogurt each day to reach his target.*

Example 5. *A fruit grower uses two types of fertilizer in an orange grove, brand A and brand B. Each bag of brand A contains* 8 *pounds of nitrogen and* 4 *pounds of phosphoric acid. Each bag of brand B contains* 7 *pounds of nitrogen and* 6 *pounds of phosphoric acid. Tests indicate that the grove needs* 720 *pounds of nitrogen and* 500 *pounds of phosphoric acid. How many bags of each brand should be used to provide the required amounts of nitrogen and phosphoric acid?*

Solution. 41 *bags of brand A and* 56 *bags of brand B.*

Section 4.2 - Systems of Linear Equations and Augmented Matrices

Matrices.

Definition 1 (Matrix)**.** *A* matrix *is a rectangular array of numbers written within brackets. The entries in a matrix are called* elements *of the matrix.*

Some examples of matrices are

$$A = \begin{bmatrix} 1 & -4 & 5 \\ 7 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & 5 & 12 & 4 \\ 0 & 1 & 8 & 3 \\ -3 & 0 & 9 & 0 \\ 7 & -9 & 22 & 10 \end{bmatrix}$$

Definition 2. *A matrix is called an* $m \times n$ matrix *if it has* m *rows and* n *columns. The expression* $m \times n$ *is called the* size *of the matrix. The numbers* m *and* n *are called the* dimensions *of the matrix. If* $m = n$*, the matrix is called a* square *matrix. A matrix with only* 1 *column is called a* column *matrix and a matrix with only* 1 *row is called a* row matrix.

For example, the matrix A above is a 2×3 matrix and the matrix B is a 4×4 matrix and so B is a square matrix.

When we write an arbitrary matrix we use the *double subscript notation*, a_{ij}, which is read as “a sub i-j”, for example, the element a_{23} is read as “a sub two-three” (not as “a sub twenty-three”); sometimes we will drop “sub” and just say "a two-three". Here is an example arbitrary $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The *principal diagonal* (or main diagonal) of a matrix is the diagonal formed by the elements $a_{11}, a_{22}, a_{33}, \ldots$. This diagonal always starts in the upper left corner, but it doesn't have to end in the bottom right. In the next examples, the principal diagonal is red.

$$A = \begin{bmatrix} {\color{red}1} & -4 & 5 \\ 7 & {\color{red}0} & -2 \end{bmatrix} \qquad B = \begin{bmatrix} {\color{red}-4} & 5 & 12 & 4 \\ 0 & {\color{red}1} & 8 & 3 \\ -3 & 0 & {\color{red}9} & 0 \\ 7 & -9 & 22 & {\color{red}10} \end{bmatrix} \qquad C = \begin{bmatrix} {\color{red}\pi} & 1 \\ 0 & {\color{red}3} \\ -7 & 6 \end{bmatrix}$$

Augmented Matrices. In this section, we will stick with systems of 2 equations. Given a system of equations

$$\begin{array}{ccccc} a_{11}x_1 & + & a_{12}x_2 & = & k_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & k_2 \end{array}$$

we have two matrices that we can associate to it, the *coefficient matrix*

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and the *constant matrix*

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}.$$

We can also put these two matrices together and form an *augmented matrix* associated to the system

$$\left[\begin{array}{cc:c} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \end{array}\right]$$

(the dashed line separates the coefficient matrix from the constant matrix).

Example 6. *Find the augmented matrix associated to the system*

$$\begin{array}{ccccc} 3x & + & 4y & = & 1 \\ x & - & 2y & = & 7 \end{array}$$

Solution.

$$\color{blue}\left[\begin{array}{cc:c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right]$$

Notation. *We will number the rows of a matrix from top to bottom and the columns of a matrix from left to right. When referring to the i^{th} row of a matrix we write R_i (for example R_2 refers to the second row) and we use C_j to refer to the j^{th} column.*

Definition 3 (Row Equivalent)**.** *We say that two augmented matrices are* row equivalent *if they are augmented matrices of equivalent linear systems. We write a $\sim$ between two augmented matrices which are row equivalent.*

This definition immediately leads to the following theorem

Theorem 1. *An augmented matrix is transformed into a row-equivalent matrix by performing any of the* row operations*:*

(a) *Two rows are interchanged ($R_i \leftrightarrow R_j$).*

(b) *A row is multiplied by a nonzero constant ($kR_i \to R_i$).*

(c) *A constant multiple of one row is added to another row ($kR_j + R_i \to R_i$).*

The arrow $\to$ is used to mean "replaces."

Solving Linear Systems Using Augmented Matrices. When solving linear systems using augmented matrices, the goal is to use row operations as needed to get a 1 for every entry on the principal diagonal and zeros everywhere else on the left side of the augmented matrix. That is, the goal is to turn in into an augmented matrix of the form

$$\left[\begin{array}{cc:c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right]$$

which corresponds to the system

$$\begin{array}{ccccc} x & & & = & m \\ & & y & = & n \end{array}$$

thus telling us that $x = m$ and $y = n$.

Example 7. *Solve the following system using an augmented matrix*

$$\begin{array}{ccccc} 3x & + & 4y & = & 1 \\ x & - & 2y & = & 7 \end{array}$$

Solution. *The augmented matrix for this system is*

$$\left[\begin{array}{cc:c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right]$$

If we subtract 3 times the second row from the first, we can get a zero in the top left:

$$\left[\begin{array}{cc:c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right] \overset{R_1 - 3R_2 \to R_1}{\sim} \left[\begin{array}{cc:c} 0 & 10 & -20 \\ 1 & -2 & 7 \end{array}\right]$$

Then we can make the second entry in the first row a 1 *by dividing the first row by* 10*:*

$$\left[\begin{array}{cc|c} 0 & 10 & -20 \\ 1 & -2 & 7 \end{array}\right] \overset{\frac{1}{10}R_1 \to R_1}{\sim} \left[\begin{array}{cc|c} 0 & 1 & -2 \\ 1 & -2 & 7 \end{array}\right]$$

Since that 1 *is in the second column, we actually want it in the second row, so switch the rows:*

$$\left[\begin{array}{cc|c} 0 & 1 & -2 \\ 1 & -2 & 7 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right]$$

Finally, we can get rid of the -2 *in the first row by adding* 2 *times the second row to the first:*

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right] \overset{R_1+2R_2 \leftrightarrow R_1}{\sim} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

So, this tells us that $x = 3$ *and* $y = -2$*.*